The equation $\mathrm{denom}(B_n) = n$ has only one solution

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Abstract

Let B_n $(n = 0, 1, 2, \ldots)$ denote the usual n-th Bernoulli number. We show that the denominator of B_n equals n if and only if $n = 1806$.

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1 Introduction

The Bernoulli numbers B_n can be defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi .$$

The numbers B_n are rational where the B_n with odd index $n > 1$ are zero and the $(-1)^{\frac{n}{2}+1} B_n$ are positive when n is even. For now, let n be an even positive integer and p denotes a prime. The denominator of B_n, see [1], is given by

$$\mathrm{denom}(B_n) = \prod_{p-1 \mid n} p \,. \tag{1.1}$$

2 Properties of the denominator of B_n

Theorem 2.1 *Let n be an even positive integer. Then*

$$\mathrm{denom}(B_n) = n \quad \Longleftrightarrow \quad n = 1806 .$$

Note that $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_n = 0$ for all odd $n > 1$. Therefore we only have to examine even indices n of B_n. Since n is an even positive integer, it easily follows that $6 \mid \mathrm{denom}(B_n)$. Equation (1.1) shows that $\mathrm{denom}(B_n)$ is a squarefree integer.

Lemma 2.2 *Let n be an even positive integer. Assume that $\mathrm{denom}(B_n) = n$. Then we have the following conditions:*

(1) $n = p_1 \cdots p_r$ *with primes* $p_1 < p_2 < \ldots < p_r$ *where* $r \geq 2$*,*

(2) $p_\nu - 1 \mid p_1 \cdots p_{\nu-1}$ *for* $\nu = 2, \ldots, r$*,*

(3) $n + 1$ *is not a prime.*

Proof. By assumption we have $\operatorname{denom}(B_n) = n$. (1): This is a consequence of (1.1) and that $6 \mid \operatorname{denom}(B_n)$. (2): We have $p_\nu - 1 \mid n$ for $\nu = 1, \ldots, r$. Since $p_1 = 2$ and $p_1 < p_2 < \ldots < p_r$, we deduce that $p_\nu - 1 \mid p_1 \cdots p_{\nu-1}$ for $\nu = 2, \ldots, r$. (3): Assume that $p = n + 1$ is a prime. Then $p - 1 \mid n$ implies that $p \mid n$. Contradiction. □

Proof of Theorem 2.1. Assume that $\operatorname{denom}(B_n) = n$. By Lemma 2.2 we have $n = p_1 \cdots p_r$ with $r \geq 2$. Since $p_1 = 2$, $p_2 = 3$, and $p_1 p_2 + 1 = 7$ is a prime, we deduce that $r \geq 3$. Now we shall construct, step by step, the prime factors of n by using Lemma 2.2.

Case $r = 3$: $n = 2 \cdot 3 \cdot p_3$. The condition $p_3 - 1 \mid 6$ only yields $p_3 = 7$, but $n = 42$ is no solution since 43 is a prime.

Case $r = 4$: $n = 2 \cdot 3 \cdot 7 \cdot p_4$. The condition $p_4 - 1 \mid 42$ only yields $p_4 = 43$. This gives a solution with $n = 1806$, since $1807 = 13 \cdot 139$ is composite.

Case $r = 5$: $n = 2 \cdot 3 \cdot 7 \cdot 43 \cdot p_5$. We have to examine the condition $p_5 - 1 \mid 2 \cdot 3 \cdot 7 \cdot 43$. This provides the possible solutions for p_5: $2 \cdot 43 + 1 = 87$, $2 \cdot 3 \cdot 43 + 1 = 259$, $2 \cdot 7 \cdot 43 + 1 = 603$, $2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807$. None of these numbers are prime. Hence, there is no solution for p_5 and n.

Since there is no solution in the case $r = 5$, it follows that there is no solution for any $r \geq 5$. This shows that $n = 1806$ is the unique solution of $\operatorname{denom}(B_n) = n$. □

References

[1] K. Ireland and M. Rosen. *A Classical Introduction to Modern Number Theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, 2nd edition, 1990.