On the $p$-divisibility of the sequence $B_{lp^r}/lp^r$.

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Let $B_n (n = 0, 1, 2, \ldots)$ be the usual $n$th Bernoulli number. For an irregular pair $(p, l)$ we define

$$\Delta_{(p,l)} \equiv p^{-1} \left( \frac{B_{l+p-1}}{l+p-1} - \frac{B_l}{l} \right) \pmod{p}$$

with $0 \leq \Delta_{(p,l)} < p$. We call $\Delta_{(p,l)}$ singular when $\Delta_{(p,l)} = 0$. Note that no singular $\Delta_{(p,l)}$ has been found yet and that $p^2 \nmid B_l$ for $p < 12\,000\,000$; see [1] for these calculations.

The following theorems are simplified reformulations of [2, Theorem 8.1, p. 434] and [2, Corollary 8.2, p. 435]. See [2, Section 8, pp. 434] for further details.

**Theorem 1** Let $p \geq 5$ be a prime and $l, r$ be positive integers where $l$ is even and $0 < l < p$. There are the following cases.

1. If $p$ is regular or $(p, l)$ is not an irregular pair, then $\text{ord}_p (B_{lp^r}/lp^r) = 0$.
2. If $(p, l)$ is an irregular pair and $p^2 \nmid B_{lp}/lp$, then $\text{ord}_p (B_{lp^r}/lp^r) = 1$.
3. If $(p, l)$ is an irregular pair and $p^2 \mid B_{lp}/lp$, then $\text{ord}_p (B_{lp^r}/lp^r) \geq 2$.

Note that no examples of case 3 of Theorem 1 are known. For a nonsingular $\Delta_{(p,l)}$, the following theorem gives a more precise result; see also [2, Section 4, pp. 415].

**Theorem 2** Let $(p, l)$ be an irregular pair where $\Delta_{(p,l)} \neq 0$. The $p$-adic zeta function $\zeta_{p,l}$ associated with $(p, l)$ has a unique simple zero $\chi_{(p,l)} \in \mathbb{Z}_p$. Then

$$\text{ord}_p (B_{lp^r}/lp^r) = 1 + \text{ord}_p \left( \chi_{(p,l)} - l p^{r-1}/p - 1 \right).$$

Assume that only the first $m$ ($m \geq 0$) $p$-adic digits of $\chi_{(p,l)}$ are equal to $l$, $a_m \neq l$:

$$\chi_{(p,l)} = l + lp + lp^2 + \cdots + lp^{m-1} + a_m p^m + \cdots.$$

Then

$$\text{ord}_p (B_{lp^r}/lp^r) = 1 + \min(r, m).$$

The case that $\chi_{(p,l)} = l + \cdots$ is equivalent to $p^2 \mid B_{lp}/lp$ where no example is known. Moreover, it seems that the $p$-adic digits of $\chi_{(p,l)}$ are randomly distributed with no regularity.

**References**
